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Rayleigh-Brillouin spectrum in special relativistic hydrodynamics

A. L. Garcia-Perciante,[1] L. S. Garcia-Colin,[2,*] and A. Sandoval-Villalbazo[3]
[1]*Departamento de Matematicas Aplicadas y Sistemas, Universidad Autonoma Metropolitana-Cuajimalpa, Artificios 40, Mexico Distrito Federal 01120, Mexico*
[2]*Departamento de Fisica, Universidad Autonoma Metropolitana-Iztapalapa, Av. Purisima y Michoacan S/N, Mexico Distrito Federal 09340, Mexico*
[3]*Departamento de Fisica y Matematicas, Universidad Iberoamericana, Prolongacion Paseo de la Reforma 880, Mexico Distrito Federal 01219, Mexico*
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In this paper we calculate the Rayleigh-Brillouin spectrum for a relativistic simple fluid according to three different versions available for a relativistic approach to nonequilibrium thermodynamics. An outcome of these calculations is that Eckart's version predicts that such spectrum does not exist. This provides an argument to question its validity. The remaining two results, which differ one from another, do provide a finite form for such spectrum. This raises the rather intriguing question as to which of the two theories is a better candidate to be taken as a possible version of relativistic nonequilibrium thermodynamics. The answer will clearly require deeper examination of this problem.

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I. INTRODUCTION

It is a well-known fact that light scattering by a simple fluid in equilibrium at a certain temperature T and pressure p is one conclusive test to verify Onsager's linear regression of fluctuations hypothesis [1], a basic assumption in classical irreversible thermodynamics [2,3]. Not only it constitutes the core behind the proof of Onsager's reciprocity theorem but it also guarantees that the equilibrium state of the fluid is stable under such fluctuations. Thus, it is legitimate to ask if the various formulations of irreversible relativistic thermodynamics so far available are at grips with such hypothesis. Even in an indirect way, this would indicate that such theories can be tested experimentally. Indeed, in the classical case one measures the so-called dynamic structure factor of the fluid $\mathcal{S}(\vec{q},\omega)$ which represents the energy scattered by the fluid from an incoming wave of wavelength λ as a function of frequency. The outcome of this measurement is the well-known Rayleigh-Brillouin (RB) spectrum [4–6]. Its central peak, the Rayleigh peak, has a width proportional to the thermal diffusivity $D_T=\kappa/\rho_0 C_V$, where κ is the thermal conductivity, ρ_0 the equilibrium density, and C_V the specific heat at constant volume. This peak represents the intensity of the thermal (entropy) fluctuations. Symmetrically located with respect to this peak there appear two peaks, the Brillouin peaks, which represent the fluctuations arising from the mechanical dissipative processes of the fluid, the sound or light absorption. They are located at $\omega=\pm C_0 k$ from the central peak, C_0 being the velocity of sound, if the probe is a sound wave. Their width is given by the famous Stokes-Kirchhoff's formula, namely,

$$\Gamma=\frac{1}{2}\left\{\frac{1}{\rho_0}\left(\frac{4}{3}\eta+\xi\right)+\frac{\gamma-1}{\gamma}D_T\right\}, \tag{1}$$

where $\gamma=C_p/C_V$ and η and ξ being the shear and bulk viscosities, respectively. The important feature here is that the precise form of this spectrum can be obtained by solving the linearized Navier-Stokes-Fourier equations of hydrodynamics for the perturbations (or fluctuations) δT, $\delta\rho$, and $\delta\vec{u}$ present in the fluid due to its microscopic structure.

Now, it may be so that this experiment per se could not be easily carried out in a laboratory in a relativistic regime for technological reasons. However, one surely must expect that the corresponding linearized equations of relativistic hydrodynamics lead to a relativistically modified spectrum which reduces to its classical counterpart in the nonrelativistic limit. This is precisely the motivation of this paper. We wish to calculate the RB spectrum for the linearized relativistic hydrodynamic equations that arise in three cases: using the Eckart-Landau Lifshitz formalism [7,8], a relativistic generalization of Meixner's theory [9], and considering the equations obtained by the authors [10,11] when the acceleration term in Eckart's theory is expressed in terms of ∇p using Euler's equations. We will refer to this case as the modified Eckart's theory.

In Secs. II–IV we shall establish the system of linearized relativistic fluid equations for the three alternatives mentioned above and analyze the modifications to the RB spectrum in each case. Section V is devoted to the discussion of the results and final remarks.

II. MEIXNER-TYPE FORMALISM

The first formalism we wish to analyze consists of the relativistic generalization of Meixner's formalism [9]. In it, the heat flux is not considered as part of the momentum-energy tensor but is included in a separate total-energy flux conservation equation. As a result, the constitutive equation for the heat flux retains its Fourier-type structure. The hydro-

*Also at El Colegio Nacional, Luis Gonzalez Obregon 23, Centro Historico, Mexico D. F. 06020, Mexico.

dynamic equations for this formalism have been obtained elsewhere [9,11,12]. The fluctuations, here denoted by a δ prefix, evolve to the equilibrium state following the linearized version of such equations. Thus, the dynamics of the fluctuations is given by

$$\delta\dot{n} + n_0 \delta\theta = 0, \tag{2}$$

$$\tilde{\rho}_0 \delta\dot{\theta} + \frac{1}{n_0 \kappa_T}\nabla^2 \delta n + \frac{\beta}{\kappa_T}\nabla^2 \delta T - A\nabla^2 \delta\theta = 0, \tag{3}$$

$$\delta\dot{T} + \frac{T_0 \beta}{n_0 c_n \kappa_T}\delta\theta - D_T \nabla^2 \delta T = 0. \tag{4}$$

The second equation is a balance equation for the longitudinal component of the fluctuations in the hydrodynamic velocity given by $\delta\theta \equiv \delta u^{\nu}_{;\nu}$, where u^{ν} is the hydrodynamic four-vector velocity. This equation is obtained by calculating the divergence of the momentum balance equation, a procedure which decouples the transverse mode whose dynamics has been already analyzed in a separate work [16]. Here n is the particle number density, T the temperature, κ_T the isothermal compressibility, β the thermal-expansion coefficient, C_n the heat capacity at constant particle density, and $A = \zeta + 4\eta/3$. We have defined $\tilde{\rho}_0 = (n_0 \varepsilon_0 + p_0)/c^2$, where ε_0 and p_0 are the internal energy and pressure, respectively. The naught subscripts denote equilibrium values, the semicolon a covariant derivative, and a colon a component of a gradient. Greek indices run from 1 to 4 and Latin ones from 1 to 3.

As mentioned above, the procedure to obtain the spectrum is the standard one and involves calculating the dispersion relation arising from the determinant of the Fourier-Laplace transformed hydrodynamic system of equations. For Eqs. (2)–(4) we obtain that

$$\begin{vmatrix} s & n_0 & 0 \\ -\frac{1}{\tilde{\rho}_0 n \kappa_T} q^2 & s + \frac{A}{\tilde{\rho}_0} q^2 & -\frac{\beta}{\tilde{\rho}_0 \kappa_T} q^2 \\ 0 & \frac{T_0 \beta}{n_0 c_n \kappa_T} & s + D_T q^2 \end{vmatrix} = 0, \tag{5}$$

which yields a cubic dispersion relation which can be written as follows:

$$s^3 + a_2 s^2 q^2 + s(a_3 q^4 + a_4 q^2) + a_5 q^4 = 0, \tag{6}$$

where the coefficients are given by

$$a_2 = \frac{A}{\tilde{\rho}_0} + D_T,$$

$$a_3 = \frac{A}{\tilde{\rho}_0} D_T,$$

$$a_4 = \frac{\gamma}{\kappa_T \tilde{\rho}_0},$$

$$a_5 = \frac{D_T}{\kappa_T \tilde{\rho}_0}, \tag{7}$$

and we have used the relation $\frac{\beta^2 T_0}{c_n n_0 \kappa_T} = \frac{c_p - c_n}{c_n} = \gamma - 1$. One can easily show that Eq. (3) has one real root given by

$$s_1 = -\frac{a_5}{a_4} q^2 \tag{8}$$

and a pair of conjugate roots

$$s_{2,3} = \left(-\frac{a_2}{2} + \frac{a_5}{2a_4}\right) q^2 \pm i\sqrt{a_4} q. \tag{9}$$

The analysis follows exactly as in the nonrelativistic case [4–6] where it is shown that Eqs. (8) and (9) are valid up to terms of order q^4. Recalling that $S(\vec{q},\omega)$ is the density-density self-correlation function, we may plot the ratio between the dynamic and static structure factors, $S(\vec{q},\omega)/S(q)$, as a function of ω for a fixed q which should yield three peaks given by the roots of the cubic equation. The mean width of the central peak, the Rayleigh peak, is determined by the real root given in Eq. (8) and thus, for the Meixner, case we obtain a width

$$\Delta_{RM} = \frac{D_T}{\gamma} q^2. \tag{10}$$

A correction due to the modified value of the thermal conductivity κ for relativistic fluids will arise and thus one expects to observe a change in the width of the peak.

The location and width of the other two peaks, the Brillouin peaks, are determined by the conjugate roots given in Eq. (9). The location of the peaks is given by the imaginary part while the width is given by the real part. In this case the doublet appears at $\omega = \pm\sqrt{a_4} q$ so that from Eqs. (7) it follows that

$$\omega_M = \pm\sqrt{\frac{\gamma}{\kappa_T \tilde{\rho}_0}} q. \tag{11}$$

One should find a shift in this location due to the relativistic value of $\tilde{\rho}_0$. The width of the doublet is given by the real part of $s_{2,3}$, that is

$$\Delta_B = \left(\frac{a_2}{2} - \frac{a_5}{2a_4}\right) q^2, \tag{12}$$

and in this case we have

$$\Delta_{BM} = \frac{1}{2}\left(\frac{A}{\tilde{\rho}_0} + \frac{\gamma - 1}{\gamma} D_T\right) q^2, \tag{13}$$

where once again, a correction due to the relativistic values of D_T and $\tilde{\rho}_0$ is expected. In both Eqs. (10) and (13) one recovers the nonrelativistic expressions when $c \to \infty$ as in this case $\tilde{\rho}_0 \to \rho_0$, the equilibrium density.

III. ECKART'S FRAMEWORK

Eckart's theory for relativistic fluids [7] is based on the construction of a momentum-energy tensor where heat flux is included. As a consequence, in order to satisfy the second law of thermodynamics, he proposed a rather controversial constitutive equation for the heat flux in which a hydrodynamic acceleration term is included. This proposal has been claimed to render the theory unphysical and motivated the use of extended theories as alternatives [13–15], as has been thoroughly discussed [10–12,16]. The aim of this section of to present the effect of such constitutive equation in the structure of the RB spectrum. Thus, once again, the starting point is the linearized set of equations for the fluctuations in a relativistic fluid, now within Eckart's theory, which are shown in Ref. [16] to read as

$$\delta\dot{n} + n_0\delta\theta = 0, \tag{14}$$

$$\tilde{\rho}_0\delta\dot{\theta} + \frac{1}{n_0\kappa_T}\nabla^2\delta n + \frac{\beta}{\kappa_T}\nabla^2\delta T - A\nabla^2\delta\theta - \frac{\kappa}{c^2}\nabla^2\delta\dot{T} - \frac{\kappa T_0}{c^4}\delta\ddot{\theta} = 0, \tag{15}$$

$$\delta\dot{T} + \frac{T_0\beta}{n_0c_n\kappa_T}\delta\theta - D_T\nabla^2\delta T - \frac{D_TT_0}{c^2}\delta\dot{\theta} = 0. \tag{16}$$

Indeed, the two terms $-\frac{\kappa T_0}{c^4}\delta\ddot{\theta}$ and $-\frac{D_TT_0}{c^2}\delta\dot{\theta}$ in Eqs. (15) and (16) come from Eckart's proposal for the heat flux constitutive equation depending on the hydrodynamic acceleration through the term $-\frac{T}{c^2}\dot{u}^\nu$ [see Eq. (19)]. All quantities appearing in these equations are the same ones as appear in Eqs. (2)–(4).

Proceeding as in the previous case, we analyze the dispersion relation which is now given by

$$\begin{vmatrix} s & n_0 & 0 \\ -\frac{1}{n\kappa_T}q^2 & -\frac{\kappa T_0}{c^4}s^2 + \tilde{\rho}_0 s + Aq^2 & \frac{\kappa}{c^2}q^2 s - \frac{\beta}{\kappa_T}q^2 \\ 0 & \frac{T_0\beta}{n_0c_n\kappa_T} - \frac{D_TT_0}{c^2}s & s + D_Tq^2 \end{vmatrix} = 0, \tag{17}$$

which yields a quartic polynomial, namely,

$$b_1s^4 + s^3 + b_2s^2q^2 + s(b_3q^4 + a_4q^2) + b_5q^4 = 0, \tag{18}$$

with the coefficients given by

$$b_1 = -\frac{\kappa T_0}{c^4\tilde{\rho}_0},$$

$$b_2 = \frac{A}{\tilde{\rho}_0} + D_T\left(1 - \frac{2\beta T_0}{c^2\kappa_T\tilde{\rho}_0}\right),$$

$$b_3 = \frac{AD_T}{\tilde{\rho}_0},$$

$$b_4 = \frac{\gamma}{\kappa_T\tilde{\rho}_0},$$

$$b_5 = \frac{D_T}{\kappa_T\tilde{\rho}_0}.$$

We now attempt a rather intuitive but accurate solution to Eq. (18). Notice that the coefficient of s^4 is very small. Neglecting it, we can readily identify three roots, namely,

$$s_1 = -\frac{b_5}{b_4}q^2,$$

$$s_{2,3} = \left(-\frac{b_2}{2} + \frac{b_5}{2b_4}\right)q^2 \pm i\sqrt{b_4}q.$$

Now, we assume that these three roots are still approximate solutions to the quartic and find the fourth root by using the property that, since the coefficient of s^3 is equal to one, $\Sigma_{i=1}^4 s_i = -\frac{1}{b_1}$, and thus

$$s_4 \simeq \frac{c^4\tilde{\rho}_0}{\kappa T_0} + \left[\frac{A}{\tilde{\rho}_0} + D_T\left(1 - \frac{2\beta T_0}{c^2\tilde{\rho}_0\kappa_T}\right)\right]q^2.$$

Since $\beta/\kappa_T < 0$, the fourth root s_4 is always positive. A real positive root in the dispersion relation yields an exponential growth in the structure factor instead of a finite spectrum. This behavior is unphysical and simply implies that the RB spectrum does not exist in Eckart's formalism even in the nonrelativistic limit.

IV. MODIFIED ECKART'S THEORY

The system of equations we consider in this section is obtained, as in the previous one, from a momentum-energy tensor which includes relativistic heat flux terms. The key difference here is that the constitutive equation introduced for the heat flux is obtained through the following argument. According to Eckart, such constitutive equation is given by

$$J^\nu_{[Q]} = -\kappa h^\nu_\mu\left(T^{,\mu} + \frac{T}{c^2}\dot{u}^\mu\right), \tag{19}$$

where the second term, as argued before [16], violates the tenets of classical irreversible thermodynamics since it is neither a thermodynamic force nor a flux. Further, it has another serious drawback, namely, it raises $\dot{u}^\nu$ to the category of a state variable, a set already chosen to be given by n, u^ν, and T. Thus, the resulting set of hydrodynamic equations would be overdetermined. Therefore, to keep Eq. (19) to first order in the gradients, we eliminate $\dot{u}^\nu$ using Euler's equation

$$\tilde{\rho}_0\dot{u}^\nu = -p_{,\mu}h^{\mu\nu}. \tag{20}$$

Now, according to the local equilibrium assumption,

$$p_{,\mu} = \frac{\beta}{\kappa_T}T_{,\mu} + \frac{1}{n_0\kappa_T}n_{,\mu}, \tag{21}$$

and thus, substitution of Eq. (21) in Eq. (19) yields

$$J^\ell_{[Q]} = -\kappa\left(1 + \frac{T_0}{c^2\tilde{\rho}_0}\frac{\beta}{\kappa_T}\right)T^{,\ell} - \frac{\kappa T_0}{n_0\kappa_Tc^2\tilde{\rho}_0}n^{,\ell} \tag{22}$$

or

$$J^{\ell}_{[Q]} = -L_{TT}T^{,\ell} - L_{nT}n^{,\ell}, \tag{23}$$

where L_{TT} is an "effective thermal conductivity" given by

$$L_{TT} = \kappa\left(1 + \frac{\beta T_0}{c^2\tilde{\rho}_0\kappa_T}\right) \tag{24}$$

and L_{nT} a new transport coefficient given by $\frac{\kappa T_0}{n_0\kappa_T c^2\tilde{\rho}_0}$ which has no classical counterpart. We would like to remark that an equation similar in structure to Eq. (23) was already derived by Landau and Lifshitz [8].

Clearly these transport coefficients can also be calculated from a kinetic model but we shall discuss such calculation in a separate paper. The equations in this formalism need not be justified in detail since they follow simply from the method used to obtain Eckart's equations incorporating the terms arising from Eq. (23) for the heat flux. Thus we obtain that

$$\delta\dot{n} + n_0\delta\theta = 0, \tag{25}$$

$$\tilde{\rho}_0\delta\dot{\theta} + \frac{1}{n_0\kappa_T}\nabla^2\delta n + \frac{\beta}{\kappa_T}\nabla^2\delta T - A\nabla^2\delta\theta - \frac{L_{TT}}{c^2}\nabla^2\delta\dot{T} - \frac{L_{nT}}{c^2}\nabla^2\delta\dot{n} = 0, \tag{26}$$

$$\delta\dot{T} + \frac{T_0\beta}{n_0c_n\kappa_T}\delta\theta - \frac{L_{TT}}{n_0c_n}\nabla^2\delta T - \frac{L_{nT}}{n_0c_n}\nabla^2\delta n = 0, \tag{27}$$

and give rise to the dispersion relation

$$\begin{vmatrix} s & n_0 & 0 \\ -\frac{1}{n_0\kappa_T}q^2 + \frac{L_{nT}}{c^2}sq^2 & \tilde{\rho}_0 s + Aq^2 & \frac{L_{TT}}{c^2}q^2 s - \frac{\beta}{\kappa_T}q^2 \\ \frac{L_{nT}}{n_0c_n}q^2 & \frac{T_0\beta}{n_0c_n\kappa_T} & s + \frac{L_{TT}}{n_0c_n}q^2 \end{vmatrix} = 0, \tag{28}$$

which can be written as

$$s^3 + d_2s^2q^2 + s(d_3q^4 + d_4q^2) + d_5q^4, \tag{29}$$

where

$$d_2 = \frac{A}{\tilde{\rho}_0} + \frac{L_{TT}}{n_0c_n}\left(1 - \frac{\beta T_0}{c^2\kappa_T\tilde{\rho}_0}\right) - \frac{n_0L_{nT}}{c^2\tilde{\rho}_0}, \tag{30}$$

$$d_3 = \frac{AL_{TT}}{n_0c_n\tilde{\rho}_0}, \tag{31}$$

$$d_4 = \frac{\gamma}{\kappa_T\tilde{\rho}_0}, \tag{32}$$

$$d_5 = \frac{1}{\kappa_T\tilde{\rho}_0 n_0c_n}(L_{TT} - \beta n_0L_{nT}). \tag{33}$$

Once more, following the steps outlined for the two previous cases, one can identify the modifications to the spectrum, as before, by analyzing the roots. For this case the width of the Rayleigh peak is given by

$$\Delta_{RS} = \frac{d_5}{d_4}q^2 = \frac{q^2}{n_0c_n\gamma}(L_{TT} - \beta n_0L_{nT}). \tag{34}$$

The shift in the Brillouin doublet is the same as in the Meixner case. In the particular case of an ideal gas, the properties of $\tilde{\rho}_0$ [17] guarantee that the position of the peaks will never exceed $\pm cq$. On the other hand, the width is significantly modified. We now obtain

$$\Delta_{BS} = \frac{q^2}{2}\left\{\frac{A}{\tilde{\rho}_0} + \frac{L_{TT}}{n_0c_n}\left(1 - \frac{1}{\gamma} - \frac{\beta T_0}{c^2\kappa_T\tilde{\rho}_0}\right) + L_{nT}\left(\frac{\beta}{c_n\gamma} - \frac{n_0}{c^2\tilde{\rho}_0}\right)\right\}. \tag{35}$$

Equations (34) and (35) deserve further attention. The former predicts a modification in Rayleigh's peak which changes its width due to the effective thermal conductivity given by Eq. (24) and the presence of L_{nT}; both, as stressed above, are strictly relativistic effects. This is quite different from Meixner-type theory where the correction arises only from the relativistic value of κ/c_n. Moreover, the shape of Brillouin's peaks is further altered due to several relativistic terms as can be seen from Eq. (35).

V. SUMMARY AND FINAL REMARKS

In the previous section, the modifications to the RB spectrum according to the three versions of relativistic irreversible thermodynamics have been explored. The difference between the modified Eckart's theory result analyzed in Sec. IV and Meixner's case analyzed in Sec. II should be emphasized. The latter one does not have a density gradient in "Fourier's equation" which, as shown in Eq. (22), is strictly a relativistic factor. This poses an intriguing question, namely, in both cases which are alternative versions of a relativistic nonequilibrium theory; we predict the existence of a RB spectrum. These spectra are different in both cases, but contain modifications in comparison to the classical spectrum and both reduce to it in nonrelativistic limit. Which theory is the correct one? If we believe in fundamentals we would be inclined to choose the version that is consistent with the results obtained from kinetic theory and thus, as we have shown in previous work [10], the second theory prevails. However, Eckart's modified theory still contains the heat flux as a component of the momentum-energy tensor which still is debatable. On the other hand if we wish a phenomenological theory that contains a density gradient in the heat flux within the Meixner-type formalism we would face the problem of how to introduce such term. The answers to these puzzles are still open. We feel that Eckart's original approach may be discarded but the final answer as to which is the appropriate version of a relativistic nonequilibrium thermodynamics is still a challenge both theoretically and experimentally. To facilitate the various results available for the RB spectrum we have summarized them in the following Table:

	Rayleigh's peak width	Brillouin peaks shift	Brillouin peaks width
Nonrelativistic	$\frac{D_T}{\gamma}q^2$	$\pm\sqrt{\frac{\gamma}{\rho_0\kappa_T}}q$	$\frac{1}{2}(\frac{A}{\rho_0}+\frac{\gamma-1}{\gamma}D_T)q^2$
Meixner	$\frac{\mathcal{D}_T}{\gamma}q^2$	$\pm\sqrt{\frac{\gamma}{\tilde{\rho}_0\kappa_T}}q$	$\frac{1}{2}(\frac{\mathcal{A}}{\tilde{\rho}_0}+\frac{\gamma-1}{\gamma}\mathcal{D}_T)q^2$
Eckart	No spectrum	No spectrum	No spectrum
Modified Eckart	$\frac{q^2}{n_0c_n\gamma}(L_{TT}-\beta n_0L_{nT})$	$\pm\sqrt{\frac{\gamma}{\tilde{\rho}_0\kappa_T}}q$	$\frac{q^2}{2}\{\frac{A}{\tilde{\rho}_0}+\frac{L_{TT}}{n_0c_n}(\frac{\gamma-1}{\gamma}-\frac{\beta T_0}{c^2\kappa_T\tilde{\rho}_0})+L_{nT}(\frac{\beta}{c_n\gamma}-\frac{n_0}{c^2\tilde{\rho}_0})\}$

where calligraphic letters are being used for relativistic transport coefficients in order to distinguish them from the nonrelativistic ones.

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